ON THE SPECTRUM OF SMALL PERTURBATIONS OF PLANE-PARALLEL COUETTE FLOW

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The stability of plane-parallel Couette flow has been investigated in a very large number of works. The aim of most of these was to prove the stability of this flow. However, the whole spectrum of normal perturbations, over a broad range of values of the Reynolds number, was apparently not sufficiently investigated, although the knowledge of the whole spectrum is necessary for the construction of a nonlinear theory of stability for Couette flow, and also for the final opinion on its stability with respect to small perturbations.

Several of the lower decrements with large values of the Reynolds number R and fixed values of the wave number α have been obtained by asymptotic methods [1]. Decrements in the case of small R were computed in [2 and 3]. The spectrum of decrements in a broad range of variation of Reynolds number can be obtained, apparently, only by numerical calculation. Recently in the paper [4], and next in [5], the results of such a calculation of the lower decrements of damped perturbations was mentioned. Later in [6] some additional data on the behavior of the four lower decrements was published.

In the present paper the results of the calculations of the decrements spectrum of normal perturbations of plane-parallel Couette flow are quoted. We will make the calculation by the method of Galerkin with a system of basis functions different from those used in [4 and 6]. An approximation containing 18 basis functions will permit us to trace, with sufficient accuracy, the behavior of the nine lower decrements and phase velocities of the normal perturbations in the region of the number αR from 0 to 1000.

1. Let us consider Couette flow between the planes $x = \pm h$ with the linear velocity profile $v_z = U_0 x$ (z is the coordinate along the flow). As units of measurement for velocity, distance, and time we will take, respectively, U_0, h, and h^2/ν (ν is the kinematic viscosity). The current function for small normal perturbations can be represented in the form $\varphi(x)\exp(-\lambda t + i\alpha z)$, where α is the real wave number and λ is the complex decrement of the perturbation. The perturbation amplitude $\varphi(x)$ satisfies the Orr-Sommerfeld equation

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R\left(x - \frac{c}{R}\right)(\varphi'' - \alpha^2\varphi)\left(R = \frac{U_0 h}{\nu},\ \lambda = i\alpha c,\ c = c_r + ic_i\right) \quad (1)$$

and boundary conditions

$$\varphi = \varphi' = 0 \qquad \text{for } x = \pm 1 \qquad (2)$$

The boundary value problem (1), (2) can be solved approximately by Galerkin's method.

For this, solution $\varphi(x)$ is represented in the form

$$\varphi(x) = \sum_{n=0}^{N} c_n \varphi_n^{(0)}(x) \tag{3}$$

As the system of basis functions $\varphi^{(0)}{}_n(x)$ it is convenient to take the complete system of normal perturbation amplitudes for the liquid at rest, i.e. the characteristic functions of the boundary value problem (1), (2) with $R = 0$ (see [3 to 8]). (This system of functions was proposed by Petrov). These functions have the form

$$\varphi_n^{(0)} = \frac{1}{\sqrt{I_n}}\left[\frac{\cosh \alpha x}{\cosh \alpha} - \frac{\cos\sqrt{\lambda_n^{(0)} - \alpha^2}\,x}{\cos\sqrt{\lambda_n^{(0)} - \alpha^2}}\right] \qquad (n = 0, 2, 4, \ldots)$$

$$\varphi_n^{(0)} = \frac{1}{\sqrt{I_n}}\left[\frac{\sinh \alpha x}{\sinh \alpha} - \frac{\sin\sqrt{\lambda_n^{(0)} - \alpha^2}\,x}{\sin\sqrt{\lambda_n^{(0)} - \alpha^2}}\right] \qquad (n = 1, 3, 5, \ldots)$$

The normalization integrals I_n and the transcendental relations for the determination of the decrements of damped perturbations $\lambda_n^{(0)}$ for the liquid at rest are given in [3]. The orthogonality conditions of Galerkin lead to the system of homogeneous algebraic equations

$$\sum_{n=0}^{N} c_n \{(\lambda - \lambda_n^{(0)})\,\delta_{mn} + i\alpha R H_{mn}\} = 0 \qquad (m = 0, 1, \ldots, N) \tag{4}$$

The matrix elements

$$H_{mn} = \int_{-1}^{1} \varphi_m^{(0)}\, x\,(\varphi_n^{(0)\prime\prime} - \alpha^2 \varphi_n^{(0)})\,dx$$

are different from zero only for indices of different parity. This permits the matrix of the system (4) to be brought to real form with the aid of a unitary transformation (for formulas for H_{mn} see [3]).

It is necessary to find the characteristic numbers λ of system (4) for fixed numbers α and R. The larger the Reynolds number and the more

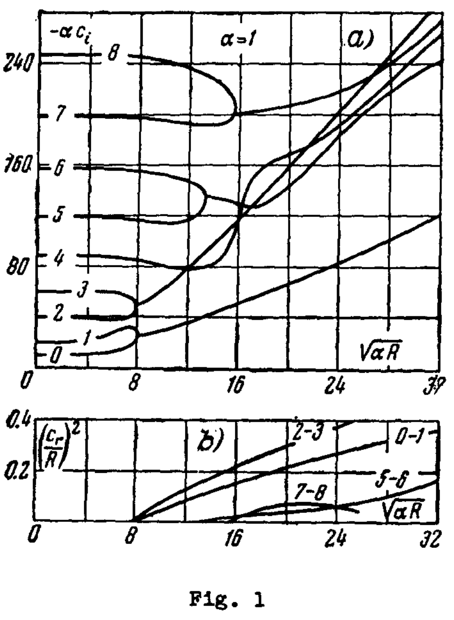

Fig. 1

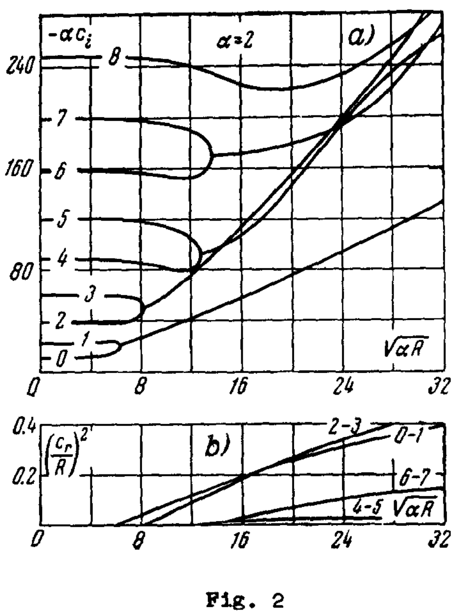

Fig. 2

spectrum levels it is necessary to find, the more functions are required in the expansion (3). In our calculation we will use 18 functions, and the problem is reduced to finding the characteristic values of a real matrix of eighteenth order. This approximation permits us to determine with good accuracy the nine lowest decrements in the domain of the number αR from 0 to 1000. In this domain the values of the decrements, obtained with $N = 16$ and $N = 17$, are practically indistinguishable. The characteristic values of the matrix are found by the orthogonal-power method [7]. The calculation of the spectrum of decrements was carried out on the ЭВЦМ «Арагац» (EVTsM "Aragats") computer in the Computing Center of the University of Perm'.

2. In the discussion of the singularities of the spectrum of decrements we consider, as an example, the spectra of perturbations with wave numbers $\alpha = 1, 2$. These spectra are presented, respectively, in Fig.1 and 2. In Fig.1a and 2a the real parts of λ, i.e. the damping decrements $-\alpha c_i$, are shown. In Fig.1b and 2b the square of the phase velocity of the perturbations is shown, measured in units of the fundamental flow speed, i.e. $(c_r/R)^2$, where $c_r = (1/\alpha)\ \mathrm{Im}\,\lambda$.

From Fig.1, which corresponds to the case $\alpha = 1$, we see that for $0 < R < 64$ all decrements λ are real and positive, i.e. the perturbations decay monotonously ($\alpha c_i < 0$), and the phase velocity of the perturbations $c_r = 0$. At $R = 64$ a confluence of the second and third monotonous levels occurs and perturbations with complex conjugate decrements appear. These perturbations have identical damping decrements $-\alpha c_i$ and their phase velocities c_r differ in sign, that is, the perturbations travel in the stream in opposite directions. For higher values of the Reynolds number we have still further singular points, at which confluences of the real levels occur with the formation of complex-conjugate pairs.

A comparison of Figs. 1 and 2 shows a strong dependance of the structure of the spectra on the wave number α. In the case $\alpha = 1$, for example, the fourth level remains real over the whole range of the number αR investigated. Its rapid variation in the domain of Reynolds numbers from 100 to 400 is due, apparently, to the strength of its interaction with some neighboring levels. For $\alpha = 2$ a confluence of the fourth and fifth levels occurs; the eighth level, however, turns out to be real. For other values of the wave number the form of the spectrum may differ from that shown in Figs. 1 and 2, but the general rules for the intersections of the perturbation decrements, formulated in [3], always hold.

A comparison of the results of the calculations with the values of the damping decrements for the lower levels, contained in [4 and 6], discloses a good correspondence. The new data, obtained in the present work, on the spectrum of perturbations for Couette flow confirms the conclusions on the stability of this flow.

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